PRINCIPLES OF ANALYSIS
LECTURE 16 - CONTINUITY

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1. Continuity

Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and let $x_0 \in E$. We say that f is *continuous at* x_0 if

$$\forall \epsilon > 0 \, \exists \delta > 0 \ni |x - x_0| \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

If f is continuous at every point in E, we say that f is *continous* (on E).

The following follows directly from previous definitions and results.

Proposition 1. *Let $E \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$ with $x_0 \in E$ and x_0 an accumulation point of E. Then the following are equivalent:*

(a) *f is continuous at x_0;*
(b) *f has a limit at x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$;*
(c) *For every sequence $\{x_n\}_{n=1}^{\infty}$ from E converging to x_0, the sequence $\{f(x_n)\}_{n=0}^{\infty}$ converges to $f(x_0)$.*

Proposition 2. *Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$. Let $x_0 \in E$. If f and g are continuous at x_0, then so are $f + g$, fg, and f/g (if $g(x_0) \neq 0$).*

Proof. Combine Proposition 1 **(c)** with the limit laws for sequences. □

Proposition 3. *Let $D, E \subset \mathbb{R}$. Let $f : D \to E$ be continuous at $x_0 \in D$ and $g : E \to \mathbb{R}$ be continuous at $y_0 = f(x_0) \in E$. Then $g \circ f : D \to \mathbb{R}$ is continuous at x_0.*

Proof. Let $\epsilon > 0$. Since g is continuous at y_0, there exists $\delta > 0$ such that

$$|y - y_0| < \delta \Rightarrow |g(y) - g(y_0)| < \epsilon.$$

In particular, for $y = f(x)$ for some x, we rewrite this as

$$|f(x) - f(x_0)| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon.$$

Since f is continuous at x_0, there exists $\gamma > 0$ such that

$$|x - x_0| < \gamma \Rightarrow |f(x) - f(x_0)| < \delta.$$

Then

$$|x - x_0| < \gamma \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon.$$

Thus $g \circ f$ is continuous at x_0. □

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2. Continuous Preimage of Open Sets

Remark 1. Let $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ and let $x_0 \in E$. Then f is continuous at x_0 if and only if We note that this is identical to the condition

$$\forall \epsilon > 0 \, \exists \delta > 0 \ni x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon).$$

Comment. This is simply a rewording of the definition. □

The next proposition implies that the continuous preimage of an open set is open.

Proposition 4. *Let $U \subset \mathbb{R}$ be open and let $f : U \to \mathbb{R}$. Then f is continuous on U if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f is continuous on U. Let $V \subset \mathbb{R}$ be open. If $f^{-1}(V) = \varnothing$ it is open, so assume that $f^{-1}(V) \neq \varnothing$, and let $x_0 \in f^{-1}(V)$. We wish to show that a neighborhood of x_0 is contained in V.

Now $f(x_0) \in V$. Since V is open, there exists $\epsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V$. Since f is continuous, there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V$. So $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open.

($\Leftarrow$) Suppose that, for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open. Let $x_0 \in U$ and let $\epsilon > 0$. Let $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$; this is an open set, so $U = f^{-1}(V)$ is open, and $x_0 \in U$, there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset U$. Then $f((x_0 - \delta, x_0 + \delta)) \subset f(U) \subset V$. Therefore $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$, which shows that f is continuous at x_0. □

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